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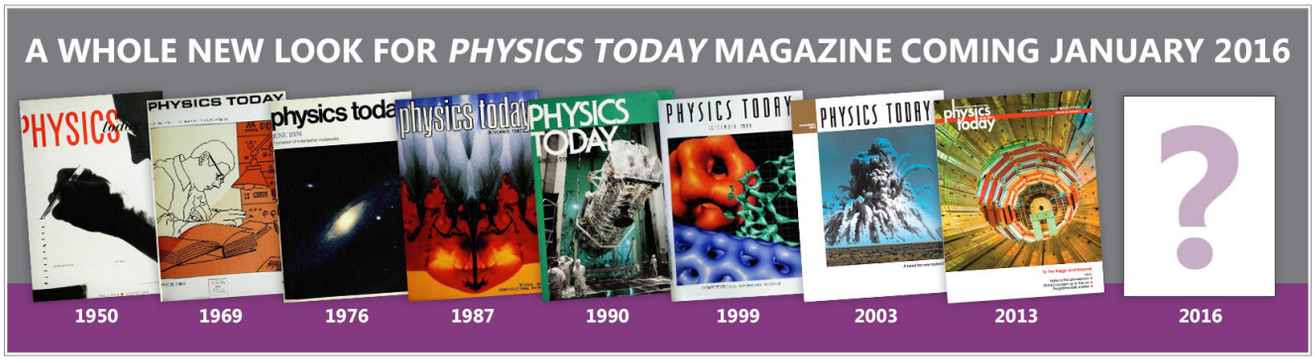

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Solving linear stochastic differential equations

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The aim of this paper is to provide the user with tools for the solution of linear differential equations with random coefficients. Only analytic methods which lead to expressions in closed form for first and second order moments and probability distributions of the solution are considered. The paper deals both with approximate methods which require the existence of a small (or large) dimensionless parameter and with the method of model coefficients, where the true coefficients of the stochastic equation are replaced by random step functions with the same first and second order moments and probability distributions, chosen in such a way that the equation can be solved analytically. The second procedure does not rely on the existence of a small parameter.

1. INTRODUCTION

Consider a linear system subject to time dependent stochastic perturbations (both in the external forces and in the parameters). The evolution of such a system is governed by a set of linear differential equations with random coefficients (stochastic equations) of the form

$$\frac{d}{dt} X_i(\omega;t) = \sum_j M_{ij}(\omega;t) X_j(\omega;t) + F_i(\omega;t), \qquad i,j = 1,\ldots,n, \quad (1.1)$$

where ω is an element of a probability space Ω, the X_i describe the state of the system in an n-dimensional space and where the parameters (coefficients) $M_{ij}(\omega;t)$ and the forces $F_i(\omega;t)$ are prescribed stationary random functions of the time variable t. To simplify the notation, ω will usually be omitted. In addition to Eq. (1.1), a set of initial conditions is given (usually nonrandom)

$$X_i(\omega;0) = X_i^0. \quad (1.2)$$

Examples of physical applications of linear stochastic differential equations are mentioned in the concluding section. Broadly speaking, by "solving" a stochastic equation we mean finding the statistical properties of the solution. Notice that most of the material covered in this paper can be extended to linear stochastic operational differential equations involving time dependent stochastic operators in an abstract finite- or infinite-dimensional space. However, the more difficult problem of stochastic partial differential equations is not covered here (see, e.g., Refs. 1–3).

When dealing with the linear stochastic equation (1.1), it is convenient to introduce the *Green's function* G satisfying an equation which in matrix notations reads

$$\frac{d}{dt} G(t,t') = M(t)G(t,t'), \qquad G(t',t') = I, \quad (1.3)$$

where I is the identity matrix. In terms of G, the solution of Eq. (1.1) with the initial condition (1.2) may be written

$$X(t) = G(t,0)X(0) + \int_0^t G(t,t')\,F(t')dt'. \quad (1.4)$$

The aim of this paper is to present the reader with a variety of methods which have proved to be useful in dealing with physical applications. We shall concentrate on analytic methods leading to exact or approximate solutions in closed form. Questions of existence, uniqueness, measurability, stability, etc., will not be considered here.[4,5,6]

It is useful to distinguish between two approaches: one either tries to find an approximate solution of the stochastic equation using the true random coefficients, or to find an exact solution using a model (e.g., a Markov process) for the random coefficients.

In Sec. 2, various approximation methods will be reviewed and their validity discussed. This includes the Born approximation, the static approximation, the Bourret and related approximations (diffusion and Hashminskii limits, Kraichnan direct interaction approximation). The concept of Kubo number, a measure of the effect of the stochastic perturbation over one correlation time, is introduced.

In Sec. 3, it is shown that the mean Green's function of a linear stochastic differential equation can be obtained explicitly for a rather large class of random coefficients called kangaroo processes (KP) for which the single time probability distribution and the two-time second order moments can be chosen in a rather arbitrary way. Particular attention is given to the validity of the approximation procedure where the true coefficients of a stochastic equation are replaced by KP coefficients.

In Sec. 4, the calculations are extended to second order moments and probability distributions of the solution, and also to the inhomogeneous case. Nonlinear stochastic differential equations are also briefly considered in connection with the Liouville equation approach. It is also shown that for certain conservative systems, the asymptotic probability distribution of the $X_i(t)$ for $t \to \infty$ can be obtained explicitly from ergodic theory.

Sections 2, 3 (excepting part C), and 4 (excepting part A) can be read independently.

Finally, we mention that, as far as the result are concerned, there is quite a bit of overlap between this paper and other papers on linear stochastic differential equations,[7,8] especially in Secs. 1B and 3A. The distinctive features of this paper are that

(i) many results usually obtained by Fokker–Planck techniques are here derived simply by averaging the equations and using the semigroup property of the Green's function;

(ii) a large class of exactly soluble equations is obtained;

(iii) the ranges of validity of the various methods are carefully examined and a guide for the user is given in the last section.

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2. APPROXIMATION PROCEDURE FOR LIMITING CASES.

A. Short time perturbation expansions: The Born and mean Born

We start from Eq. (1. 3) for the Green's function, written as

$$\frac{d}{dt}\, G(t,t') = [M_0 + M_1(t)]G(t,t'), \quad G(t',t') = I, \tag{2. 1}$$

where we have separated the stationary random matrix $M(t)$ into its mean value M_0 and its fluctuating part $M_1(t)$; Eq. (2. 1) is easily recast into the following integral form

$$G(t,t') = e^{M_0(t-t')} + \int_{t'}^{t} e^{M_0(t-t'')} M_1(t'') G(t'',t')\, dt'' \tag{2. 2}$$

which, when iterated, yields the well-known von Neumann series

$$\begin{aligned} G(t,t') = {} & e^{M_0(t-t')} + \int_{t'}^{t} dt_1\, e^{M_0(t-t_1)} M_1(t_1)\, e^{M_0(t_1-t')} \\ & + \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, e^{M_0(t-t_1)} M_1(t_1)\, e^{M_0(t_1-t_2)} \\ & \times M_1(t_2)\, e^{M_0(t_2-t')} + \cdots . \end{aligned} \tag{2. 3}$$

To study the convergence of this expansion, we assume that M_0 and $M_1(t)$ are operators acting in a normed space. The norm of the vector X is denoted $\| X \|$. Furthermore, we assume that

$$\| e^{M_0 t} \| \leqq 1. \tag{2. 4}$$

This condition is satisfied if, e.g., M_0 is anti-Hermitian or dissipative. In the rest of this paper we shall denote by σ the order of magnitude of the fluctuations of the coefficients of the stochastic equation (1. 1). This can be measured, e.g., by the largest dispersion of the coefficients of $M(t)$ assumed to be finite. To avoid unnecessary complications, we assume in this section, the much stronger condition

$$\| M_1(t) \| \leqq \sigma \tag{2. 5}$$

almost surely and for any t. It is then easily seen that the norm of the nth term in the perturbation expansion (2. 3) is less than

$$|t-t'|^n \ \frac{\sigma^n}{n!}.$$

We conclude that the perturbation expansion is always convergent and that

$$G(t,t') = e^{M_0(t-t')} + O(|t-t'|\,\sigma); \tag{2. 6}$$

for moderate values of $|t-t'|\,\sigma$ we can use the *Born approximation*

$$\begin{aligned} G(t,t') = {} & e^{M_0(t-t')} + \int_{t'}^{t} e^{M_0(t-t_1)} M_1(t_1)\, e^{M_0(t_1-t')}\, dt_1 \\ & + O[(|t-t'|\,\sigma)^2]. \end{aligned} \tag{2. 7}$$

Consider now the mean Green's function $\langle G(t,t')\rangle$. Since $\langle M_1(t)\rangle = 0$, the second term in Eq. (2. 3) vanishes upon averaging. Expanding to second order we obtain the "mean Born approximation"

$$\begin{aligned} \langle G(t,t')\rangle \approx {} & e^{M_0(t-t')} + \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, e^{M_0(t-t_1)} \\ & \times \langle M_1(t_1)\, e^{M_0(t_1-t_2)} M_1(t_2)\rangle\, e^{M_0(t_2-t')}. \end{aligned} \tag{2. 8}$$

At first sight, the validity of (2. 8) as an approximation still requires $|t-t'|\,\sigma \ll 1$. However, let us assume that $M_1(t)$ has a finite correlation time, i.e., that its autocorrelation is integrable; define the correlation time T_{corr} as the integral scale of the autocorrelation [roughly speaking, T_{corr} is the time over which $M_1(t_1)$ and $M_1(t_2)$ are appreciably correlated]. Now, we notice that the major contribution to the double integral in (2. 8) comes from $|t_1 - t_2| \lesssim T_{\mathrm{corr}}$; as a consequence the order of magnitude of the second term on the rhs of (2. 8) is only $\sigma^2 T_{\mathrm{corr}}\, |t-t'|$ and not $\sigma^2\, |t-t'|^2$. Hence, the validity of the mean Born approximation requires

$$\sigma^2\, T_{\mathrm{corr}}\, |t-t'| \ll 1, \tag{2. 9}$$

which is weaker than $|t-t'|\,\sigma \ll 1$ provided that $|t-t'| \gg T_{\mathrm{corr}}$.

B. Weak perturbations: Bourret approximation, the white noise, and Hashminskii limits

Clearly, when $|t-t'|\,\sigma \gg 1$, the perturbation expansion is of little use. We now seek an approximate expression for $\langle G(t,t')\rangle$ valid for arbitrarily large $|t-t'|$.

Iterating the integral equation (2. 2) once averaging, we obtain

$$\begin{aligned} \langle G(t,t')\rangle = {} & e^{M_0(t-t')} + \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, e^{M_0(t-t_1)} \\ & \times \langle M_1(t_1)\, e^{M_0(t_1-t_2)} M_1(t_2)\, G(t_2,t')\rangle. \end{aligned} \tag{2. 10}$$

We notice that

$$\langle G(t,t')\rangle = \langle G(t-t',0)\rangle \tag{2. 11}$$

which is a consequence of the stationarity of $M_1(t)$; we may therefore as well set $t' = 0$. Differentiating with respect to t, we obtain

$$\begin{aligned} \frac{d}{dt}\langle G(t,0)\rangle = {} & M_0\langle G(t,0)\rangle + \int_0^t \langle M_1(t)\, e^{M_0(t-t')} \\ & \times M_1(t')\, G(t',0)\rangle\, dt'. \end{aligned} \tag{2. 12}$$

Bourret[9] has proposed the following closure approximation

$$\begin{aligned} & \langle M_1(t)\, e^{M_0(t-t')} M_1(t')\, G(t',0)\rangle \\ & \quad \approx \langle M_1(t)\, e^{M_0(t-t')} M_1(t')\rangle \langle G(t',0)\rangle, \end{aligned} \tag{2. 13}$$

originally obtained by him as a first order approximation on the basis of a diagrammatic expansion;[9] this approximation can also be obtained quite differently as will be shown below.

Equation (2. 12) reduces upon use of (2. 13) to a simple integrodifferential equation for $\langle G(t,0)\rangle$ which we shall call the *Bourret equation:*

$$\begin{aligned} \frac{d}{dt}\langle G(t,0)\rangle = {} & M_0\langle G(t,0)\rangle + \int_0^t \langle M_1(t)\, e^{M_0(t-t')} \\ & \times M_1(t')\rangle \langle G(t',0)\rangle\, dt', \quad G(0,0) = I. \end{aligned} \tag{2. 14}$$

Equivalent equations have been proposed by Keller[10] and Frisch [2]; closed equations of this type for mean quantities are generally called *master equations*. Notice that the Bourret equation is easily solved by Laplace transformation. Indeed, defining

$$\langle \tilde{G}(z)\rangle = \int_0^\infty e^{izt} \langle G(t,0)\rangle\, dt \tag{2. 15}$$

and

$$\tilde{K}(z) = \int_0^\infty e^{izt} \langle M_1(t)\, e^{M_0 t} M_1(0)\rangle\, dt, \tag{2. 16}$$

we obtain

$$\langle \tilde{G}(z)\rangle = [-iz - M_0 - \tilde{K}(z)]^{-1}. \tag{2. 17}$$

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Now we investigate the validity of the closure assumption (2. 13). Let us assume that

$$\| G(t, 0) \| \leqq 1 . \tag{2. 18}$$

This condition is usually satisfied, since in most applications $M_0 + M_1(t)$ is anti-Hermitian and, hence, $G(t, 0)$ is unitary. It is known that any Green's function satisfies a semigroup property

$$G(t', 0) = G(t', s)\, G(s, 0). \tag{2. 19}$$

From the preceding section, we have

$$G(t', s) = e^{M_0(t'-s)} + O(\sigma(t' - s)). \tag{2. 20}$$

From (2. 18), (2. 19), and (2. 20) we obtain

$$G(t', 0) = e^{M_0(t'-s)}\, G(s, 0) + O(\sigma(t' - s)). \tag{2. 21}$$

Assuming

$$\sigma \mid t' - s \mid \ll 1, \tag{2. 22}$$

we can write the lhs of (2. 13) in the following form:

$$\langle M_1(t)\, e^{M_0(t-t')}\, M_1(t')\, G(t', 0)\rangle \approx \langle M_1(t)\, e^{M_0(t-t')} M_1(t') \times e^{M_0(t'-s)}\, G(s, 0)\rangle . \tag{2. 23}$$

We now make the fundamental assumption that $M_1(t)$ has a finite correlation time T_{corr}. For

$$\mid t' - s \mid \gg T_{\mathrm{corr}} \tag{2. 24}$$

the stochastic Green's function $G(s, 0)$, which is a functional of $M_1(\tau)$ for $0 < \tau < s$, is only very weakly correlated to $M_1(t')$ and $M_1(t)(t > t')$. It is therefore legitimate to factorize the rhs of (2. 23) to obtain

$$\langle M_1(t)\, e^{M_0(t-t')}\, M_1(t')\, G(t', 0)\rangle \approx \langle M_1(t)\, e^{M_0(t-t')}\, M_1(t')\rangle \times \langle e^{M_0(t'-s)}\, G(s, 0)\rangle . \tag{2. 25}$$

Using again (2. 20), we obtain the desired closure approximation (2. 13).

The compatibility of (2. 22) and (2. 24) obviously requires

$$K = \sigma\, T_{\mathrm{corr}} \ll 1 . \tag{2. 26}$$

The dimensionless number K, sometimes called the generalized Reynolds number,[2] will be called the *Kubo number*, because it was first introduced by Kubo.[11]

In deriving the Bourret equation we implicitly assumed $t \gg T_{\mathrm{corr}}$; this is indeed a consequence of $t > t' > s > 0$ and of (2. 24). In fact, the Bourret equation is also valid for small times since it can be checked that the perturbation expansion solution of Eq. (2. 14) agrees with the mean Born approximation (2. 8) up to the order of σ^2.

It is interesting to notice that the closure approximation (2. 13) and the Bourret equation become exact, whatever the Kubo number, if $M_1(t)$ is of the form

$$M_1(t) = m(t)\, L_1, \tag{2. 27}$$

where $m(t)$ is a dichotomic Markov process (also called random telegraph process) and L_1 is a constant matrix.[12] Recall that the dichotomic Markov process is defined as a step function with values ± 1, the transitions occurring at Poisson distributed times; this process is a special case of the KAP introduced in Sec. 3A.

The Bourret equation (2. 14) is a "non-Markovian" master equation, i.e., the derivative of $\langle G(t, 0)\rangle$ involves an integral over past values of the mean Green's function. Yet, the Bourret equation can be used as starting point for the derivation of various Markovian approximations which we shall now consider.

For $t \gg T_{\mathrm{corr}}$ the Bourret equation can be reduced to the following Markovian form, first given by Kubo:[11]

$$\frac{d}{dt}\langle G(t, 0)\rangle = \left(M_0 + \int_0^\infty \langle M_1(s)\, e^{M_0 s}\, M_1(0)\, e^{-M_0 s}\rangle\, ds\right) \times \langle G(t, 0)\rangle, \quad \langle G(0, 0)\rangle = I. \tag{2. 28}$$

To derive Eq. (2. 28) from the Bourret equation (2. 4), we notice that, as a consequence of $K \ll 1$, we have for $\mid t - t' \mid < T_{\mathrm{corr}}$

$$G(t'0) \approx e^{-M_0(t-t')}\, G(t, 0). \tag{2. 29}$$

To obtain (2. 28) we then put $t - t' = s$ and integrate over s from zero to infinity, rather than from zero to t; this is legitimate provided that the covariance of $M_1(t)$ is integrable, since the integrand will be negligible for $t \gg T_{\mathrm{corr}}$.

The Kubo equation (2. 28) has two limiting cases which actually cover all situations as we shall find later. First, the *white noise limit:* write $M_1(t) = \sigma M_1'(t/T_{\mathrm{corr}})$ and let $T_{\mathrm{corr}} \to 0, \sigma \to \infty$ in such a way that $\sigma^2\, T_{\mathrm{corr}} \to D$. It is easily seen that in this limit the factors $e^{\pm M_0 s}$ in Eq. (2. 28) cancel out and that the Kubo equation goes over into

$$\frac{d}{dt}\langle G(t, 0)\rangle = M_0 \langle G(t, 0)\rangle + D \int_0^\infty \langle M_1'(s)\, M_1'(0)\rangle\, ds \langle G(t, 0)\rangle . \tag{2. 30}$$

Since, in the white noise limit, the Kubo number $K = \sigma T_{\mathrm{corr}}$ goes to zero, Eq. (2. 30) becomes exact. (Notice that, whereas the amplitude of white noise is infinite, its strength, measured by the Kubo number, is zero.) In Ref. 13 the reader will find another derivation of a master equation equivalent to (2. 30) which uses the fact that white noise can be defined as the limit of shot noise.

We turn now to the *Hashminskii limit*. If we let the strength σ of the stochastic perturbation go to zero, the variations of the Green's function over a finite time interval will be entirely due to M_0. We now factor out the variation to M_0 by introducing the "interaction representation"

$$\langle G(t, 0)\rangle = e^{+M_0 t}\, \langle G_I(t, 0)\rangle . \tag{2. 31}$$

Then, we let $t \to \infty$ in such a way that $\sigma^2 t$ remains finite; this results in a finite variation of $\langle G_I\rangle$. Indeed, writing $M_1 = \sigma M_1'$ and $t = \tau/\sigma^2$, we find that in the limit $\sigma \to 0$, the Kubo equation (3. 28) goes over into the Hashminskii equation

$$\frac{d}{d\tau}\, \langle G_I(\tau, 0)\rangle = H \langle G_I(\tau, 0)\rangle , \tag{2. 32}$$

wherein

$$H = \lim_{\sigma \to 0}\, e^{-M_0 \tau/\sigma^2} \int_0^\infty \langle M_1'(s)\, e^{M_0 s}\, M_1'(0)\, e^{-M_0 s}\rangle\, ds\, e^{+M_0 \tau/\sigma^2}. \tag{2. 33}$$

Limits of the form $\lim_{\sigma \to 0} e^{-M_0 \tau/\sigma^2} A e^{+M_0 \tau/\sigma^2}$ are frequently used in the quantum mechanical theory of S matrices.[14] The existence of the limit requires that M_0 be anti-Hermitian; it is then easily checked that H commutes with M_0 (hint: take a representation where M_0 is diagonal). This result greatly simplifies the resolution

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of the Hashminskii equation.[15] Other derivations of the Hashminskii equation, based on Fokker–Planck techniques may be found in Refs. 16 and 17.

Let us now investigate more closely the validity of the white noise equation (2. 30) and the Kubo equation (2. 28); we see that the only difference is the drop out of the factors $e^{\pm M_0 s}$. Since the integral over s extends over roughly one correlation time $T_{\rm corr}$, we may safely neglect the exponentials if the following condition is fulfilled

$$\| M_0 \| \, T_{\rm corr} \ll 1. \tag{2. 34}$$

For the Hashminskii limit the problem is somewhat more difficult. Consider the Kubo equation (2. 28); the first operator M_0 on the rhs, which is usually anti-Hermitian, does not contribute to the relaxation of the mean Green's function as $t \to \infty$. This relaxation comes entirely from the second operator

$$R = \int_0^\infty \langle M_1(s) e^{M_0 s} M_1(0) e^{-M_0 s} \rangle \, ds. \tag{2. 35}$$

By dimensional analysis we find that this operator is of the order of $\sigma^2 T_{\rm corr}$. Hence, the relaxation time $t_{\rm rel}$ must be of the order of $(\sigma^2 T_{\rm corr})^{-1}$. If we now rewrite the Kubo equation in the interaction representation, we obtain

$$\frac{d}{dt} \langle G_I(t, 0) \rangle = e^{-M_0 t} R e^{+M_0 t} \langle G_I(t, 0) \rangle. \tag{2. 36}$$

If t is large enough we can replace $e^{-M_0 t} R e^{+M_0 t}$ by its limit for $t \to \infty$ which is precisely the Hashminskii operator H (within trivial changes of notations). Now the times t of interest are of the order of $t_{\rm rel} \approx (\sigma^2 T_{\rm corr})^{-1}$, hence the condition to be fulfilled is

$$\| M_0 \| \, (\sigma^2 T_{\rm corr})^{-1} \gg 1. \tag{2. 37}$$

If we recall that $K = \sigma T_{\rm corr} \ll 1$, we find that one of the two conditions (2. 34) and (2. 37) is automatically satisfied; there is even some overlap. We thus arrive at the important conclusion that if the Kubo number is small and if $t \gg T_{\rm corr}$ it is always possible to use one of the two white noise and Hashminskii limits.

Remark: Kraichnan[18] has proposed another master equation for the mean Green's function called the direct interaction equation. With

$$\frac{d}{dt} \langle G(t, 0) \rangle = M_0 \langle G(t, 0) \rangle + \int_0^t \langle M_1(t) \langle G(t - t', 0) \rangle \times M_1(t') \rangle \langle G(t', 0) \rangle \, dt'. \tag{2. 38}$$

This nonlinear equation is an exact consequence of Kraichnan's random coupling model. It can also be obtained from the theory of parastochastic operators.[3] The usefulness of the Kraichnan equation as an approximation is questionable. For small values of the Kubo number, the Kraichnan equation is equivalent to the much simpler Bourret equation, and for large values of the Kubo number, it can no longer be used as an approximation. For linear stochastic ordinary differential equation, a much more powerful method will be described in the subsequent sections. Nevertheless, the Kraichnan method of stochastic models, the essence of which is to introduce an additional stochastic element into the equation to make it tractable, remains very useful in dealing with nonlinear stochastic equations, particularly in the field of turbulence.[19-21]

C. Strong perturbations: The static approximation

The Bourret equation is limited to the case when the effect of the stochastic perturbation over one correlation time is weak, i.e., when $\sigma T_{\rm corr} \ll 1$. In the opposite case,

$$K = \sigma T_{\rm corr} \gg 1, \tag{2. 39}$$

the mean Green's function $\langle G(t, 0) \rangle$ can be substantially affected by the stochastic perturbation for times t satisfying

$$t \ll T_{\rm corr}. \tag{2. 40}$$

Since the stochastic operator $M(\tau)$ undergoes insignificant changes for $0 \leqq \tau \leqq t \ll T_{\rm corr}$, we may as well neglect its time dependence (but not its randomness) and integrate Eq. (1. 3) to obtain

$$\langle G(t, 0) \rangle \approx G_S(t) = \langle \exp\{tM\} \rangle. \tag{2. 41}$$

$G_S(t)$ will be called the static mean Green's function. It is often useful to deal with the static resolvent, the Laplace transform of the static Green's function

$$\tilde{G}_S(z) = \int_0^\infty e^{izt} G(t) \, dt = \langle \{ -iz - M \}^{-1} \rangle. \tag{2. 42}$$

The explicit calculation of $\tilde{G}_S(z)$ requires only a matrix inversion and an averaging over the *probability distribution* of the coefficients of M.

At first sight, the static approximation is restricted to $t \ll T_{\rm corr}$. However, in many problems, the mean Green's function is damped by phase mixing or dissipation for $t \to \infty$. If the damping time is small compared to $T_{\rm corr}$, the static Green's function can be used to describe the full relaxation. An example is provided by the pseudo-oscillator discussed in Sec. 3C.

3. THE METHOD OF MODEL COEFFICIENTS

From the preceding chapter, we know that the Born approximation is limited to short times and that the Bourret and static approximation are limited to respectively small and large Kubo numbers. Such approximations are of no use if the Kubo number is of the order of one and if it is necessary to follow the evolution of the mean Green's function over times long enough so that there is an appreciable damping by phase mixing. There is thus need for a method which puts no restriction on the Kubo number; of course, the results of Sec. 2 should be recovered in the corresponding limits.

If we recall that the Bourret equation involves in an essential way the two time second order moments (covariance) of the stochastic coefficients, whereas the static Green's function involves the single-time probability distribution of the coefficients, it is clear that an approximate master equation for the mean Green's function should involve both the probability distribution and the covariance. It turns out that it is possible to construct a class of stepwise constant Markovian random functions with arbitrary probability distributions and rather arbitrary covariances. Such functions, when used as coefficients in a linear stochastic equation, lead to a closed analytic expression for the mean Green's function.

A. The Kubo-Anderson process (KAP)

The Kubo–Anderson process (KAP) is a stepwise constant random function which jumps at randomly chosen times between random step-values. The times

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$t_1, t_2, \ldots$ will be called jumping times. A more precise definition will be given below. The KAP has been introduced in connection with nuclear magnetic resonance[22,23] and introduced again, in a special case, by Bourret[24] as a tool for linear stochastic equations (see also Ref. 25).

Definition: The step-wise constant random function $m(t)$ is called a Kubo–Anderson process (KAP) if the jumping times $t_i\,(i = -\infty, \ldots, +\infty)$ are uniformly and independently distributed in $(-\infty, +\infty)$ with density ν (Poisson distribution) and $m(t)$ is a constant, $m(t) = m_i$, for $t_i \leqq t < t_{i+1}$; the m_i' are independent random variables with the same probability density $P(m)$.

We notice that $m(t)$ is a stationary Markov process with probability density $P(m)$. Assuming $\langle m \rangle = 0$ for simplicity, we obtain for the covariance of $m(t)$

$$\langle m(t)\, m(t') \rangle = \langle m^2 \rangle\, e^{-\nu |t-t'|}. \tag{3.1}$$

We see that for a KAP the probability density $P(m)$ and the correlation time

$$T_{\text{corr}} = \nu^{-1} \tag{3.2}$$

may be chosen arbitrary, but not the functional form of the covariance, which is always exponential.

We shall now show that the linear stochastic equation

$$\frac{d}{dt}\, G(t,0) = M(t)\, G(t,0), \qquad G(0,0) = I, \tag{3.3}$$

can be solved analytically for the mean Green's function provided that the coefficients of the stochastic matrix are KAP's with all the same jumping-times or, in short, when $M(t)$ is a KAP.

If there is no jumping-time between 0 and t, $M(\tau)$ remains constant for $0 \leqq \tau \leqq t$ and we are back to the static case (Sec. 2C). The probability of this event is $e^{-\nu t}$ and the corresponding contribution to $\langle G(t,0) \rangle$ is

$$\begin{aligned} \langle G(t,0) \rangle_{\text{no jump}} &= e^{-\nu t} \langle e^{tM} \rangle \\ &= e^{-\nu t}\, G_S(t). \end{aligned} \tag{3.4}$$

In the opposite case, let t' denote the last jumping-time before t. It is a well-known property of the Poisson process that the probability for this jump to occur between t' and $t' + dt'$ is $\nu e^{-\nu(t-t')}\, dt'$. Using the semigroup property of the Green's function, $G(t,0) = G(t,t')\, G(t',0)$, we can write the corresponding contribution to $\langle G(t,0) \rangle$ as

$$\langle G(t,0) \rangle_{\text{jumps}} = \int_0^t \nu e^{-\nu(t-t')} \langle G(t,t')\, G(t'0) \rangle_{t'} dt', \tag{3.5}$$

where $\langle \cdot \rangle_{t'}$, is a conditional average knowing that a jump occurred at t'. Using the fact that M remains constant between t' and t and that its value is independent of the values of $M(\tau)$, $0 \leqq \tau \leqq t'$, and therefore also independent of $G(t',0)$ which is a functional of $M(\tau)$, we obtain

$$\langle G(t,t')\, G(t',0) \rangle_{t'} = G_S(t-t') \langle G(t',0) \rangle_{t'}. \tag{3.6}$$

We claim that

$$\langle G(t',0) \rangle_{t'} = \langle G(t',0) \rangle. \tag{3.7}$$

Indeed, the knowledge that a jump occurred at t' imposes no constraints on previous jumping-times and previous values of $M(\tau)$. Adding the contribution of (3.4) and (3.5) and using (3.6) and (3.7), we obtain the Kubo–Anderson master equation

$$\begin{aligned} \langle G(t,0) \rangle_{\text{KAP}} = G_S(t)\, e^{-\nu t} + \nu \int_0^t e^{-\nu t(t-t')}\, G_S(t-t') \\ \times \langle G(t',0) \rangle_{\text{KAP}}\, dt'. \end{aligned} \tag{3.8}$$

This master equation can be solved for the mean resolvent, the Laplace transform of the mean Green's function,

$$\langle \tilde{G}(z) \rangle_{\text{KAP}} = \int_0^\infty e^{izt} \langle G(t,0) \rangle_{\text{KAP}}\, dt. \tag{3.9}$$

The solution reads

$$\langle \tilde{G}(z) \rangle_{\text{KAP}} = [I - \nu \tilde{G}_S(z + i\nu)]^{-1}\, \tilde{G}_S(z + i\nu), \tag{3.10}$$

where the static resolvent $\tilde{G}_S(z)$ is given by Eq. (3.3). Another derivation of the master equation may be found in Ref. 25 where the KAP is called the Poisson step process.

Equation (3.10) constitutes a strikingly simple result: the KAP resolvent is an algebraic function of the static resolvent; it turns out that in many applications, the quantity of interest is the resolvent and not the Green's function itself.[25,26]

Remark: The KAP can be slightly generalized to include the case of jumping-times selected according to a compound Poisson process, i.e., when the density $\nu(t)$ is a (deterministic) function of the time. A straightforward modification of Eq. (3.8) yields

$$\begin{aligned} \langle G(t,0) \rangle = G_S(t) \exp\Big(-\int_0^t \nu(\tau)\, d\tau\Big) + \int_0^t dt'\, \nu(t') \\ \times \exp\Big(-\int_{t'}^t \nu(\tau)\, d\tau\Big)\, G_S(t-t') \langle G(t',0) \rangle. \end{aligned} \tag{3.11}$$

This equation may be useful in the study of nonstationary processes.

B. The kangaroo process (KP)

We recall that a KAP has an exponential covariance. The study of the problem of stochastic Stark broadening,[25] where the covariance is proportional to $1/t$ and is not even integrable, has led us to modify the KAP by requiring that the frequency of jumping times is a function $\nu(m)$ of the value of the process itself. The new process is called a "kangaroo process" (KP).

Definition: A KP is a step-wise constant Markov process[27] with stationary transition probability given for infinitesimal time intervals by

$$\begin{aligned} P_{t\tau}(m, \Delta t \mid m', 0) = \{1 - \nu(m')\Delta t\}\, \delta(m' - m) \\ + \nu(m')\, \Delta t\, Q(m), \end{aligned} \tag{3.12}$$

where $Q(m)$ is a given probability density.

$P_{t\tau}\, dm$ is the probability that the kangaroo process at time Δt is between m and $m + dm$ knowing that it was equal to m' at time 0. The meaning of Eq. (3.12) is clear: $\{1 - \nu(m')\, \Delta t\}$ is the probability that no jump occurred in the time interval $(0, \Delta t)$ and $\nu(m')\, \Delta t$ the probability that at last one jump occurred. Immediately after such a jump, the probability density of m becomes $Q(m)$. We stress the fact that $Q(m)$ is not the stationary probability density of $m(t)$. Indeed, the Fokker–Planck equation[27] for the kangaroo process reads

$$\begin{aligned} \frac{\partial}{\partial t}\, P(m,t) &= \lim_{\substack{\Delta t \to 0 \\ \Delta t \geqq 0}} \{ \int P_{t\tau}(m, \Delta t \mid m', 0)\, P(m',t)\, dm' - P(m,t) \} / \Delta t \\ &= -\nu(m)\, P(m,t) + Q(m) \int \nu(m')\, P(m',t)\, dm'. \end{aligned} \tag{3.13}$$

Hence, the stationary probability density $P(m)$ of $m(t)$ is related to $Q(m)$ by

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$$Q(m) = \nu(m)\,P(m)/\int \nu(m')\,P(m')\,dm' = \nu(m)\,P(m)/\langle\nu\rangle. \tag{3.14}$$

Next, we evaluate the covariance of a KP. The calculation of $\Gamma(t) = \langle m(t)\,m(0)\rangle$ requires the summation of a series to take into account the possible occurrence of an arbitrary number of jumps between 0 and t. After some algebra we obtain for the Laplace transform

$$\tilde{\Gamma}(z) = \int_0^\infty e^{izt}\,\Gamma(t)\,dt, \tag{3.15}$$

the following result

$$\tilde{\Gamma}(z) = \left\langle \frac{m^2}{\nu(m) - iz} \right\rangle_S - \frac{1}{iz\langle \nu(m)/[\nu(m) - iz]\rangle_S} \left\langle \frac{m}{\nu(m) - iz} \right\rangle_S^2. \tag{3.16}$$

In many cases, e.g., if $P(m)$ and $\nu(m)$ are *even*, we have

$$\langle m/(\nu(m) - iz)\rangle_S = 0 \quad \text{or, equivalently,} \quad \langle me^{-\nu(m)t}\rangle_S = 0;$$

then, the above result simplifies to

$$\tilde{\Gamma}(z) = \left\langle \frac{m^2}{\nu(m) - iz} \right\rangle_S \tag{3.17}$$

or

$$\Gamma(t) = \int_{-\infty}^{+\infty} m^2 e^{-\nu(m)|t|}\,P(m)\,dm\,, \tag{3.18}$$

which is just an ordinary variance conditioned by the probability $e^{-\nu(m)t}$ that no jump occurs between 0 and t. The interesting point, about formula (3.18) is that it can easily be inverted: Given $P(m)$ and the covariance $\Gamma(t)$, the jumping frequency $\nu(m)$ can be calculated as follows. Assume that $\nu(m)$ is a monotonic increasing function of $|m|$ such that $\nu(\infty) = \infty$; this is a reasonable assumption since in most applications very strong values of the stochastic perturbations last only for a very short time. Taking ν as new integration variable, we obtain

$$\Gamma(t) = 2\int_{\nu(0)}^\infty m^2 P(m)\,\frac{dm}{d\nu}\,e^{-\nu|t|}\,d\nu \tag{3.19}$$

which is essentially a Laplace integral. A calculation of $\nu(m)$ requires the inversion of the Laplace transformation and the solution of a simple differential equation. An example may be found in Ref. 25 (Sec. 5). In connection with this inversion, Table I gives some useful results. Notice that the inversion is not always feasible. Indeed, from (3.18) we see that the derivative of the covariance is necessarily discontinuous at the origin; we do not know whether this is a sufficient condition for inversion. We conclude that it is always possible to construct a KP with an arbitrary probability distribution and a (quite) arbitrary covariance.

A linear stochastic equation with KP coefficients can again be solved implicitly for the mean Green's function. Indeed, the KP resolvent can be put in the following form:

TABLE I. Kangaroo process: expression of the jumping frequency $\nu(m)$ in terms of the one-time probability distribution $P(m)$ for several types of covariance $\Gamma_{KP}(t)$.

To obtain $\Gamma_{KP}(t) = \langle m(t)m(0)\rangle_{KP} =$	Use $\nu(m) =$	Remarks
$e^{-\nu_0\|t\|}$	ν_0	
$\dfrac{\Gamma(1/n)}{\|t\|^{1/n}}$	$\left(2\int_0^{\|m\|} m^2P(m)\,dm\right)^n$	$\Gamma(\cdot)$ is the gamma function. $P(m)$ even
$\sigma^2/(1 + \|t\|)$	$-\log\left(2\int_{\|m\|}^\infty m^2P(m)\dfrac{dm}{\sigma^2}\right)$	$P(m)$ even
$\sigma^2\dfrac{1 - \|t\|e^{-\pi\|t\|/2}}{1 + t^2}$	$\arccos\left(2\int_{\|m\|}^\infty m^2P(m)\dfrac{dm}{\sigma^2}\right)$	$P(m)$ even
$\sigma^2\dfrac{\|t\| + e^{-\pi\|t\|/2}}{1 + t^2}$	$\arcsin\left[2\int_0^{\|m\|} m^2P(m)\dfrac{dm}{\sigma^2}\right]$	$P(m)$ even

$$\langle\tilde{G}(z)\rangle_{KP} = \langle\tilde{G}\rangle_S + \langle\nu\tilde{G}\rangle_S\{\langle\nu(I - \nu\tilde{G})\rangle_S\}^{-1}\langle\nu\tilde{G}\rangle_S, \tag{3.20}$$

where ν denotes $\nu(M)$,

$$\tilde{G} = \{[\nu(M) - iz]\,I - M\}^{-1}, \tag{3.21}$$

and $\langle\cdot\rangle_S$ denotes the averaging over the stationary distribution of M (static averaging). Again, the resolvent is expressed in terms of purely static quantities. A proof of Eq. (3.20) may be given which parallels the proof for the KAP given in Sec. 3A. Another more constructive proof will be found in Ref. 25, Sec. 4.

Remark 1: In some applications the stochastic operator $M(t)$ appears naturally as the sum of two (or more) processes with quite different correlation times. To deal with these situations, a *compound KP* has been defined and the corresponding mean resolvent has been calculated (cf., Ref. 25, Sec. 6).

Remark 2: The KP has a non-Markovian generalization which allows an arbitrary probability distribution for the step-length, still conditioned by the step-value of M. Let

$$B(M, t) = \text{prob}\{t_{i+1} - t_i \geqq t \mid M(\tau) = M \text{ for } t_i < \tau < t_{i+1}\} \tag{3.22}$$

be the conditional probability distribution of any step-length. Let $Q(M)$ and $P(M)$ denote respectively the probability density of step-values and of the stationary process $M(t)$; they are related by

$$Q(M) = \frac{P(M)}{\left\{\int_0^\infty B(M,\tau)\,d\tau \quad \langle 1/\int_0^\infty B(M,\tau)\,d\tau\rangle_S\right\}}, \tag{3.23}$$

where $\langle\cdots\rangle_S$ denotes averaging over $P(M)$. With these notations the resolvent is again expressible in closed form as

$$\langle\tilde{G}(z)\rangle = L_z\left(\int_t^\infty C(M,\tau)\,d\tau\right) + L_z[C(M,t)] \times \left\{L_z\left[\left(\delta(t) + \frac{d}{dt}\right)C(M,t)\right]\right\}^{-1} L_z[C(M,t)] \tag{3.24}$$

with

$$C(M,t) = B(M,t)/\int_0^\infty B(M,\tau)\,d\tau \tag{3.25}$$

and

$$L_z[f(M,t)] = \int_0^\infty dt e^{izt}\langle e^{Mt}f(M,t)\rangle_S. \tag{3.26}$$

C. The method of model coefficients used as approximation

We have seen in Sec. 2 that for weak (resp. strong) perturbations, the mean Green's function of a linear stochastic equation depends essentially on the covariance (resp. the probability density) of the coefficients. The question naturally arises how close the KP solution will fit the true solution when the true coefficients are replaced by KP's with the same probability distributions and covariances.

Let us first check that the KP solution (3.20) is in agreement with the true solution for short times satisfying $t_\sigma \ll 1$. To the lowest nontrivial order, the mean Green's function is then given by the mean Born approximation (2.8) which involves only the covariance of $M_1(t)$. Hence, for short times, the mean Green's function depends only on the covariance of $M_1(t)$.

We show now that the KP solution (3.20) reduces indeed to the previously obtained approximations of Sec. 2 for very weak and very strong perturbations. We recall that the strength of perturbations is measured by

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$K = \sigma T_{\text{corr}} \approx \sigma/\langle\nu\rangle$, where σ is the order of magnitude of the fluctuating part of $M(t)$. We thus obtain the limit of very strong perturbations by letting $\nu(M) \to 0$. In this limit the KP resolvent (3. 20) reduces obviously to $G_S(z)$, which is the static resolvent; this is in agreement with the result of Sec. 2C.

To study the opposite limit, we write

$$M(t) = M_0 + \sigma M_1'(t) \quad \text{and} \quad \nu(M) = T_{\text{corr}}^{-1}\, \nu'(M_1'). \tag{3. 27}$$

As in Sec. 2B when deriving the white noise limit, we let $T_{\text{corr}} \to 0$, $\sigma \to \infty$, $\sigma^2 T_{\text{corr}} \to D$, and obtain from equation (3. 20)

$$\langle \tilde{G}(z)\rangle_{\text{KP}} = [-izI - M_0 - D\langle M_1'^2/\nu'(M_1')\rangle]^{-1} \quad \text{(diffusion limit).} \tag{3. 28}$$

This is equivalent to the following master equation

$$\frac{d}{dt}\langle G(t,0)\rangle_{\text{KP}} = (M_0 + D\int_0^\infty \langle M_1'(\tau)\, M_1'(0)\rangle_{\text{KP}}\, d\tau)\langle G(t,0)\rangle_{\text{KP}} \tag{3. 29}$$

which is identical with the white noise limit (2. 30).

Remark: The proof given here that the KP is correct in the limit of weak perturbations relies implicitly on the assumption that the covariance of $M(t)$ is integrable. In Refs. 25 and 28, we have checked, in a special case, that the KP can still be used as an approximation when the covariance is proportional to $1/t$.

So far we have only checked the agreement between the KP solution and the exact solution in limiting cases. In intermediate range of moderate perturbations, it is very difficult to draw any general conclusions. Special cases have been investigated which show indeed very good agreement; in particular, when the KP is applied to Stark broadening of spectral lines.[25-28] As a quantitative test for the validity of the intermediate range, we have compared the true and KP solutions for a randomly frequency modulated pseudo-oscillator satisfying the scalar equation

$$\frac{d}{dt}\, g(t,0) = im(t)g(t,0), \tag{3. 30}$$

where $m(t)$ is a real zero mean value stationary Gaussian process with covariance $\langle m(t)\, m(t')\rangle = \sigma^2 e^{-\nu|t-t'|}$.

Equation (3. 30) has an exact solution[2]

$$\langle g(t,0)\rangle = \exp\{-K^2[t_1 + e^{-t_1} - 1]\} \tag{3. 31}$$

with $K = \sigma/\nu$ and $t_1 = \nu t$. The corresponding exact resolvent can be written as

$$\langle \tilde{g}(z)\rangle_E = e^{K^2} \sum_{n=0}^{\infty} \frac{(-K^2)^n}{n!(n - iz + K^2)}. \tag{3. 32}$$

The KP resolvent for the same problem reduces to a KAP resolvent, since the covariance of $m(t)$ is an exponential.

The KP resolvent reads

$$\langle \tilde{g}(z)\rangle_{\text{KP}} = \left(\int \frac{-iz - iKm}{1 - iz - iKm}\, e^{-m^2/2}\, dm\right)^{-1} \cdot \int \frac{e^{-m^2/2}\, dm}{1 - iz - iKm}. \tag{3. 33}$$

The real parts of $\langle \tilde{g}(z)\rangle_E$ and $\langle \tilde{g}(z)\rangle_{\text{KP}}$ for z real, i.e., the Fourier transform of $\langle g(t,0)\rangle_E$ and $\langle g(t,0)\rangle_{\text{KP}}$, have been calculated numerically for different values of the Kubo number K and plotted as a function of z: results are shown on Fig. 1. For small and large values of K, the agreement is almost perfect. In the intermediate range $K \approx 1$ the discrepancy is at most 10%.

D. Stochastic equations with Markovian and shot noise coefficients.

The KAP and the KP processes constitute special cases of Markov processes. A general theory can be given for linear stochastic differential equations with Markovian coefficients,[2] based on the fact that the joint process $\{M(t), G(t,0)\}$ is also a Markov process; this leads to a Fokker–Planck or a Chapman–Kolmogorov equation for the joint probability density. This method should not be recommended since it leads to rather complicated partial differential or integral equations for which closed analytic solutions are generally not available.

Another case worth mentioning has been considered by Blume.[29] The coefficients are taken in the form of shot noise

$$M(t) = M_0 + \sum_{-\infty}^{+\infty} M_i\, \delta(t - t_i), \tag{3. 34}$$

where the t_i's are Poisson-distributed with density ν and the M_i's are independent identically distributed random matrices. This case, which is very similar to the KAP, leads to the following resolvent:

$$\langle G(z)\rangle = [-iz - M_0 + \nu(I - \langle \exp M\rangle)]^{-1}. \tag{3. 35}$$

4. CALCULATION OF VARIOUS STATISTICAL QUANTITIES

A. Simultaneous and time-displaced second-order moments

By second-order moments, we understand the quantities $\langle X_i(t)\, X_j(t)\rangle$ or, in short, $\langle X(t) \otimes X(t)\rangle$. We notice that, for zero, the right-hand side in Eq. (1. 1) we have

$$\langle X(t) \otimes X(t')\rangle = \langle G(t,0) \otimes G(t',0)\rangle\, (X^0 \otimes X^0), \tag{4. 1}$$

by definition of the tensor product of two matrices.

Let us first consider the case of simultaneous moments, i.e., $t = t'$. We introduce the *double Green's function*

$$\mathcal{G}(t,t') = G(t,t') \otimes G(t,t'). \tag{4. 2}$$

Differentiating (4. 2) with respect to t and using the fundamental stochastic equation (1. 3), we find that $\mathcal{G}(t,t')$, satisfies another linear stochastic equation, namely

$$\frac{d}{dt}\, \mathcal{G}(t,t') = \mathfrak{M}(t)\, \mathcal{G}(t,t'), \qquad \mathcal{G}(t,t') = I, \tag{4. 3}$$

where

$$\mathfrak{M}(t) = M(t) \otimes I + I \otimes M(t). \tag{4. 4}$$

Clearly, Eq. (4. 3) is a linear stochastic equation of the standard form. Hence, the calculation of *simultaneous* second-order moments has been reduced to the calculation of first-order moments.[30,31]

The calculation of time-displaced moments of the form $\langle G(t,0) \otimes G(t',0)\rangle$ is somewhat more involved, except when the random coefficients are of white noise type (i.e., have zero correlation time). Indeed, assuming that $t \geqq t'$, which is no loss of generality, and using the semigroup property, we have

$$\langle G(t,0) \otimes G(t',0)\rangle = \langle G(t,t')\, G(t',0) \otimes G(t',0)\rangle. \tag{4. 5}$$

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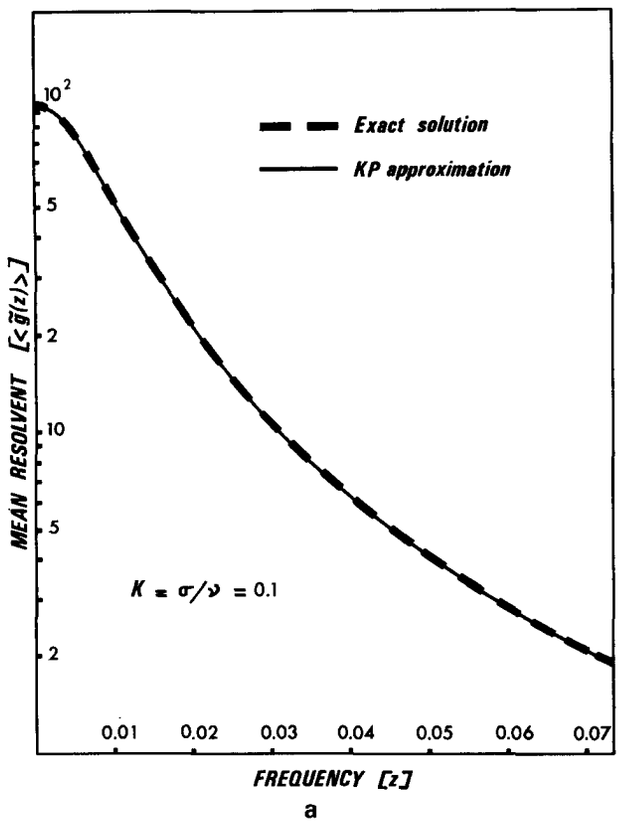

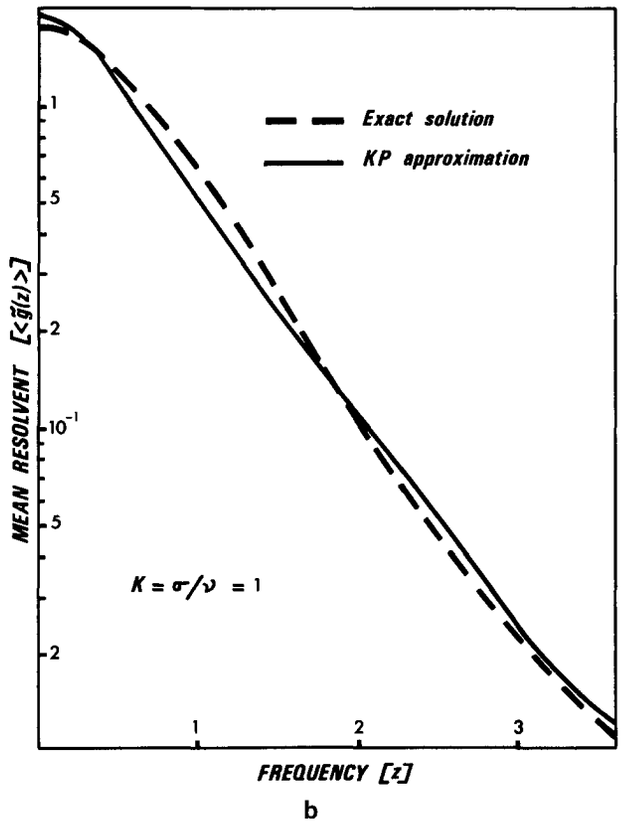

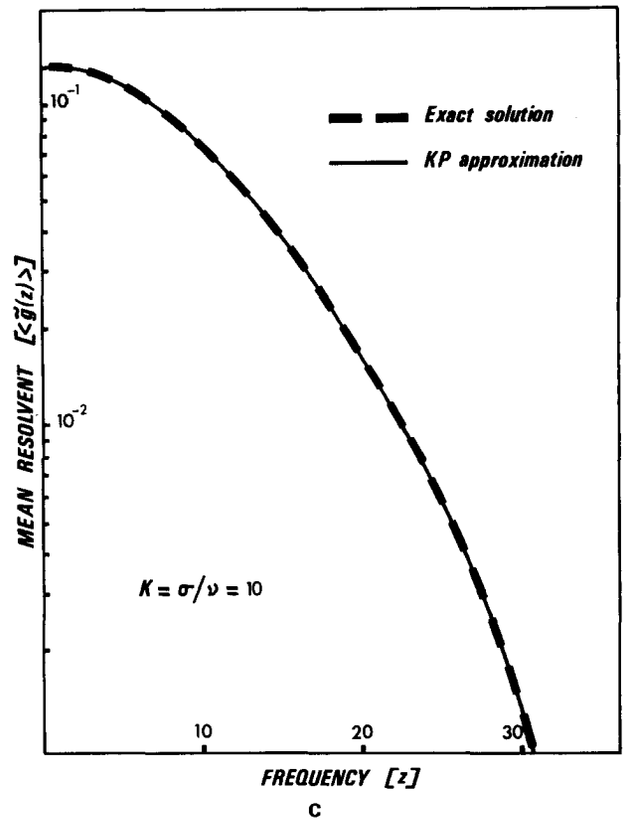

FIG. 1. *Comparison of the exact mean resolvent and of its KP approximation for the randomly modulated pseudo-oscillator defined in Section 7.*

a: $K = \sigma/\nu = 0.1$: the agreement is perfect.
b: $K = \sigma/\nu = 1$: the discrepancy is at most 10%.
c: $K = \sigma/\nu = 10$: the agreement is perfect.

Using the independence of $G(t,t')$ and $G(t',0)$ which are functionals of $M(\tau)$ for $t \geqq \tau \geqq t'$ and $t' \geqq \tau \geqq 0$, respectively, we finally obtain

$$\langle G(t,0) \otimes G(t',0)\rangle = \langle G(t-t',0)\rangle \langle \mathcal{G}(t',0)\rangle, \qquad (4.6)$$

which is a shorthand notation for $(\langle G\rangle \otimes I)\langle \mathcal{G}\rangle$.

The general case can also be dealt with if we assume that $M(t)$ is a KAP. The essential idea is to notice that the KAP is a Markov Process. It follows that if $M(t')$ is given, the past ($\tau < t'$) and the future ($\tau > t'$) become independent. If we define the conditional mean Green's functions $\langle G(t,t';M)\rangle_c$ and $\langle \mathcal{G}(t,t';M)\rangle_c$, conditioned by $M(t') = M$, we obtain, using (4.6),

$$\langle G(t,0) \otimes G(t',0)\rangle = \langle \langle G(t,t';M)\rangle_c \langle \mathcal{G}(t,t';M)\rangle_c \rangle_M, \qquad (4.7)$$

where $\langle\cdot\rangle_M$ denotes an averaging over the probability distribution of M (static averaging). It remains to calculate the conditional mean Green's function. First, we notice that from stationarity

$$\langle G(t,t';M)\rangle_c = \langle G(t-t',0;M)\rangle_c. \qquad (4.8)$$

Denoting by $\langle \tilde{G}(z;M)\rangle_c$ and $\langle \tilde{\mathcal{G}}(z;M)\rangle_c$ the Laplace transforms of $\langle G(t,0;M)\rangle_c$ and $\langle \mathcal{G}(t,0;M)\rangle_c$, we easily obtain, using the same method and notations as in Sec. 2C

$$\langle \tilde{G}(z;M)\rangle_c = [I - \nu \tilde{G}_S(z+i\nu)]^{-1}(-iz+\nu-\mathfrak{M})^{-1} \qquad (4.9)$$

and, similarly,

$$\langle \tilde{\mathcal{G}}(z;M)\rangle_c = [I - \nu \mathcal{G}_S(z+i\nu)]^{-1}(-iz+\nu-\mathfrak{M})^{-1}. \qquad (4.10)$$

This method, which is somewhat reminiscent of a method introduced by Morrison and McKenna[30,31] may lead to rather tedious calculations. A much simpler method is described in Ref. 11 for the special case when $M(t)$ is of the form

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$$M(t) = M_0 + m(t) L_1,$$

where $m(t)$ is a dichotomic Markov process (random telegraph process).

B. Inhomogeneous equations

In general, because of dissipation, the solution of a linear stochastic equation with initial conditions and no rhs relaxes to zero as $t \to \infty$ (there is, however, a notable exception for conservative systems, see Sec. 4C). A nonzero stationary solution may be obtained in the presence of a random rhs (random driving forces). In this section we shall therefore be concerned with the inhomogeneous case

$$\frac{d}{dt} X_i(t) = \sum_j M_{ij}(t) X_j(t) + F_i(t); \quad i,j = 1,\ldots,n. \tag{4.11}$$

$M_{ij}(t)$ is a stationary random matrix as before, and the $F_i(t)$ constitute a set of stationary random functions independent of the $M_{ij}(t)$'s. For the sake of simplicity we shall assume that $F_i(t)$ is a real zero mean value white noise, i.e.,

$$\langle F_i(t)\rangle = 0, \quad \langle F_i(t) F_j(t')\rangle = S_{ij}\,\delta(t-t'), \tag{4.12}$$

where S_{ij} is a constant positive definite matrix.

In order to get stationary solutions we assume that $M_{ij}(t)$ has a dissipative part, so that

$$\lim_{t\to\infty} G_{ij}(t,t') = 0, \tag{4.13}$$

where $G_{ij}(t,t')$ is the random Green's function defined by (1.3). In terms of the Green's function we may write the solution of (4.11) as

$$X_i(t) = \sum_j G_{ij}(t,0) X_j^0 + \int_0^t \sum_j G_{ij}(t,t') F_j(t')\,dt'. \tag{4.14}$$

We shall be interested in the statistical properties of $X(t)$ for $t \to \infty$, and especially in the first- and second-order moments.

Taking the average of (4.14) and using (4.12) and (4.13), we obtain

$$\lim_{t\to\infty} \langle X_i(t)\rangle = 0. \tag{4.15}$$

Let us now evaluate the time displaced second-order moment,

$$\Gamma_{ij}(\tau) = \lim_{t\to\infty} \langle X_i(t) X_j(t+\tau)\rangle. \tag{4.16}$$

From (4.14), we obtain, using (4.13) and the independence of the Green's function and the driving forces,

$$\Gamma_{ij}(\tau) = \lim_{t\to\infty} \int_0^t \int_0^{t+\tau} \sum_{nm} \langle G_{in}(t,t') G_{jm}(t+\tau,t'')\rangle \times \langle F_n(t') F_m(t'')\rangle\,dt'\,dt''. \tag{4.17}$$

Finally, using (4.12) and the fact that

$$\langle G_{in}(t,t') G_{jm}(t+\tau,t')\rangle = \langle G_{in}(s,0) G_{jm}(s+\tau,0)\rangle, \quad s = t - t', \tag{4.18}$$

we obtain

$$\Gamma_{ij}(\tau) = \int_0^\infty \sum_{n,m} \langle G_{in}(s,0) G_{jm}(s+\tau,0)\rangle\,ds\, S_{nm}, \tag{4.19}$$

which is the desired result. The calculation has been reduced to that of the second-order moment of the Green's function (see preceding section).

In the same context another (obvious) result is worth mentioning: The mean Green's function is the inter-correlation function of the solution and the right-hand side when $S_{ij} = \delta_{ij}$,

$$\lim_{t\to\infty} \langle X_i(t+\tau) F_j(t)\rangle = \langle G_{ij}(\tau)\rangle. \tag{4.20}$$

The main interest of the results (4.19) and (4.20) is that $\Gamma_{ij}(\tau)$ and $\lim_{t\to\infty}\langle X_i(t+\tau) F_j(t)\rangle$ are easily measurable quantities using a time average over a single realization, whereas the mean Green's function is not directly measurable.

C. Probability distributions: The Liouville equation

In this section, we shall consider the following problem: Let there be given the *nonlinear* stochastic differential equation

$$\frac{d}{dt} X_i(\omega;t) = A_i[m(\omega;t); X_j], \quad j = 1,\ldots,n, \tag{4.21}$$

where A_i is a nonlinear real deterministic function of the real (scalar or vector-valued) random function $m(\omega;t)$ and of X_j $(j = 1,\ldots,n)$. In addition, we assume deterministic real initial conditions X_j^0.

Our purpose is to evaluate the joint probability density $P(t;X_1,\ldots,X_n)$ of $X_1, X_2, \ldots, X_n$ at time t. We shall use the so-called "Liouville equation" method which reduces the present problem to a linear stochastic equation of type studied in previous sections.

For each realization $m(\omega;t)$, let us denote by $X_i(\omega;t)$ the solution of Eq. (4.21). We shall assume existence and uniqueness. Let us think of $X_i(\omega,t)$ as a point which, starting from X_i^0, moves around in a stochastic fashion in an n-dimensional phase space. Introducing the "fine grained density,"

$$\rho(\omega;t;X_1,\ldots,X_n) = \delta\{X_1 - X_1(\omega;t)\}\cdots\delta\{X_n - X_n(\omega;t)\}, \tag{4.22}$$

we notice that the joint probability P is the average of the fine grained density, i.e.,

$$P(t;X_1,\ldots,X_n) = \langle \rho(\omega;t;X_1,\ldots,X_n)\rangle \tag{4.23}$$

which follows immediately from (4.22).

From the equation of motion (4.21), we may derive an equation of continuity, or Liouville equation, for the fine grained density, which reads

$$\frac{\partial}{\partial t}\rho + \sum_{i=1}^n \frac{\partial}{\partial X_i}[A_i\{m(\omega;t), X_i\}\rho] = 0. \tag{4.24}$$

To derive the Liouville equation (4.24), let us introduce an indefinitely differentiable test function $\varphi(X_1,\ldots,X_n)$. Integrating the lhs of (4.24) after multiplication by φ, we obtain, using (4.22),

$$\begin{aligned}\int\left(\frac{\partial\rho}{\partial t} + \sum_1^n \frac{\partial}{\partial X_i}[A_i p]\right)\varphi\,dX_1\ldots dX_n &= \frac{d}{dt}\int \rho\varphi\,dX_1\ldots dX_n - \int \sum_1^n \rho A_i \frac{\partial\varphi}{\partial X_i}\,dX_1\ldots dX_n \\ &= \sum_{i=1}^n \frac{dX_i}{dt}\frac{\partial\varphi[X_j(t)]}{\partial X_i} - \sum_{i=1}^n A_i \frac{\partial\varphi[X_j(t)]}{\partial X_i};\end{aligned} \tag{4.25}$$

this quantity vanishes identically because of (4.21). Since this property holds for an arbitrary test function, we have proved (4.24).

In spite of the fact that it contains partial derivatives, the Liouville equation can be treated as a stochastic ordinary differential equation since the random function

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$m(\omega;t)$ does not depend on $X_1,\ldots,X_n$. The calculation of the joint probability distribution can now be carried out using the same methods as described in previous sections.

D. Asymptotic behavior of linear conservative systems from ergodic theory

Let us come back again to the linear stochastic equation

$$\frac{d}{dt}X = M(\omega;t)X, \qquad X(0) = X_0. \tag{4.26}$$

We assume now that X is a vector and M is a random matrix in a real n-dimensional space, and that equation (4.26) is *conservative* in the sense that

$$\sum_{i=1}^{n} X_i^2(t) = \text{const}, \tag{4.27}$$

This is obviously equivalent to the requirement that $M(\omega;t)$ be antisymmetric. For convenience, we shall assume that

$$\|X_0\|^2 = \sum_{i=1}^{n} |X_i(0)|^2 = 1. \tag{4.28}$$

The phase space of the system under consideration is then the n-dimensional unit sphere S.

Since $M(t)$ is antisymmetric, the mapping $X(t') \to X(t)$ is unitary. Therefore, the motion on the unit sphere S preserves the uniform measure dm. This situation is reminiscent of a problem in classical statistical mechanics: Given a system of interacting classical particles enclosed in a box, it is known that the point representing the system in the phase space remains on the energy surface, that the motion on the energy surface has an invariant measure, and that the point will eventually fill up the whole energy surface with a density proportional to the invariant measure, provided that a certain condition of metrical transitivity is satisfied.[32] In statistical mechanics, these results are proved by means of the Birkhoff ergodic theorem.[33]

Because of these similarities we expect that, under certain conditions to be specified later, the point $X(\omega;t)$ will eventually fill up the whole "energy surface" $\|X\| = 1$ with a uniform density. We do not want, here, to go into the mathematical details of the ergodic theory of the stochastic equations. We shall just state the main conditions to be satisfied by $M(\omega;t)$, and the results.

TABLE II. Range of validity of various approximations for the mean Green's function.

Approximation and equation number	Condition on Kubo number $K = \sigma T_{corr}$	Condition on t	Further condition	Remarks (See Sec. 4E)
Born (2.7)		$\sigma t \ll 1$		
Mean Born (2.8)	$K<1$ $K>1$	$K\sigma t \ll 1$ $\sigma t \ll 1$		
Bourret (2.14)	$K \ll 1$			(1)
White noise (2.30)		$t \gg T_{corr}$	$\|M_0\| T_{corr} \ll 1$	
Hashminskii (2.32)			$\|M_0\| T_{corr} \gg K^2$	
Static (2.41)		$t \ll T_{corr}$		(2)
KAP (3.10) and KP (3.20)				(3)

We require that

(i) $M(\omega;t)$ is a stationary ergodic (matrix-valued) random function of t; roughly, this means that ensemble averages of functionals of $M(\omega;t)$ are equal to time averages.

(ii) For any X_0 on the unit sphere S, and any set A of positive measure on the unit sphere, the probability that a solution of (4.26), starting from X_0, will never penetrate A is zero (stochastic metric transitivity).

It may then be shown that, for any function $f(X)$ which is measurable with respect to the uniform measure dm on the unit sphere,

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t f[X(\omega;\tau)]d\tau \underset{\text{a.s.}}{=} \left(\int_S f[X]dm\right)/\left(\int_S dm\right) = \lim_{t\to\infty}\langle f[X(\omega;t)]\rangle. \tag{4.29}$$

As an illustration of this result, let us calculate the asymptotic values of the first and second moment of $X_i(\omega;t)$, assuming that dm is normalized.

For the first-order moment we obtain

$$\lim_{t\to\infty}\langle X_i(\omega;t)\rangle = \int_S X_i dm = 0 \tag{4.30}$$

from a symmetry argument.

For the second-order moments we obtain

$$\lim_{t\to\infty}\langle X_i^2(\omega;t)\rangle = \int_S X_i^2 dm = \frac{1}{n}\int_S (X_1^2+\cdots+X_n^2)dm = \frac{1}{n}\int_S dm = \frac{1}{n}, \tag{4.31}$$

and

$$\lim_{t\to\infty}\langle X_i(\omega;t)X_j(\omega;t)\rangle = 0 \quad \text{if} \quad j\neq i. \tag{4.32}$$

It is interesting to notice that the asymptotic distribution of the "energy" $\langle X_i^2\rangle$ is simply equipartition. This has interesting applications to the energy transfer between randomly coupled oscillators.[34]

Remark: The above ideas may sometimes be extended to nonlinear stochastic equations if there is an invariant measure. For an application to the stochastic Ricatti equation

$$\frac{dz}{dx} + z^2 + n^2(\omega;x) = 0. \tag{4.33}$$

which is encountered in the theory of wave propagation in a one-dimensional random medium, the reader is referred to Ref. 35.

E. A guide for the user

We give now a few practical indications for the user who wants to calculate the mean Green's function of a linear stochastic equation. Calculation of other statistical quantities are usually reducible to the former as we have seen in Sec. 4.

First check if the equation falls into one of the classes of exactly soluble equations: white noise, shot noise, KAP, KP, Markovian coefficients, etc. If it does, the equation is usually soluble in closed analytic form except for the case of Markovian coefficients where the solution of a Fokker–Planck equation is required. If not, some approximation procedure must be used. Then, separate the stochastic evolution operator into its mean part M_0 and its fluctuating part $M_1(t)$. Estimate the correlation time T_{corr} and the dispersion of $M_1(t)$ and the norm $\|M_0\|$. Then, evaluate the dimensionless Kubo number $K = \sigma T_{corr}$. Recall that K is a measure of the effect of the stochastic perturbation over one

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correlation time. According to the values of K Table II indicates the optimal method(s) for each case. A number of remarks on this table are in order.

(1) If the stochastic perturbation is a dichotomic Markov process [c f. Eq. (2. 27)] the Bourret equation (2. 14) is exact. Its solution by Laplace transformation usually requires only a little algebra. Notice that the Bourret equation may also be considered as a first semiquantitative approach to any stochastic equation by suitably adjusting the dispersion and correlation time of a dichotomic Markov process.

(2) The static approximation describes the full relaxation of the mean Green's function only if $K \gg 1$.

(3) The KAP and KP approximations should be used when a wide range of Kubo numbers is involved (including $K \approx 1$). If a good accuracy is wanted (e.g., in line broadening problems) and if the covariance is a suitable candidate for the inversion problem, use the KP method. If not, in particular, if the covariance of the random coefficients is not too well known, use the KAP method with

$$\nu = T_{\mathrm{corr}}^{-1} .$$

To conclude, we stress that the KAP and KP methods are probably the most flexible tools presently available when it is required to solve a linear stochastic differential equation over a large range of values of the Kubo number. In contrast to most approximate methods in mathematical physics, they do not rely on the existence of a small expansion parameter.

Applications of the methods described in this paper to problems of physical interest are discussed in other papers. Among the possible fields of application, let us mention: Stark broadening,[25,28] line formation in turbulent stellar atmospheres,[26] stability and Brownian motion of linear and nonlinear dynamical systems with random parameters,[12] energy transfer between randomly coupled dynamical systems,[7,34] and propagation of waves in a one-dimensional random medium.[2,7,30,31,35-38]

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